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Empirical models – metal fatigue

S-N curve
Standard case

Reversed loading,
\[ R = \frac{\sigma_{\text{min}}}{\sigma_{\text{max}}} = -1 \]

Question: How does \( \sigma_{\text{fat}} \) change with \( R \)?
Answer: Let’s do experiments and find out.
Experiments and models

Typical S-N data for steel

Proposed models (there are more)
Constant Life Diagram (CLD)

Note: Widely different Predictions by models

⇒ No choice but to do testing, a lot of it!
Not convinced?: Look at Comparison with experimental data – not good!
Constant life diagrams for short fiber composites (E-glass/polyamide 6,6) Data: Mallick and Zhou, IJF, 2004
Plotting without models
Constant life diagrams for glass/epoxy (wind turbine materials) (from Nijssen, 2006)

Note:
“Predictions” sensitive to how much S-N data are used

At best, these diagrams provide overview of how fatigue life changes with mean stress (R ratio)

Note the asymmetry in tension (right of $R = -1$) vs compression (left of $R = -1$) due to mechanism differences
Phenomenological models

The "phenomenon" of fatigue is assumed to reflect itself in some measurable property, whose rate of change (degradation) can be taken as a function of applied cyclic loading parameters.

\[ \frac{d(\text{property})}{dN} = f(\sigma_a, R, \ldots) \]

Property: strength, stiffness, dissipated energy, entropy, etc.
Phenomenological models – metal fatigue

**Case 1** (Axles, thick parts without stress concentrations)
- Crack initiation dominates
- No measurable property change

**Case 2** (Welded parts, sharp notches, corners, etc.)
- Crack propagation dominates
- (Residual) strength changes
Phenomenological models – metal fatigue

Strength degradation

Crack Size, $a$

Residual Strength, $RS$

$RS = \frac{K_c}{\lambda \sqrt{a}}$

$K_c$: fracture toughness

$\lambda$: crack/part geometry factor

$N_d$ Number of cycles

$N_f$ Time in service, $t$

$2a$
Fatigue life prediction from strength degradation

\[ \frac{da}{dN} = C(\Delta K)^n \]

Integrate from \( a_0 \) to \( a_c \) and use

\[ RS = \frac{K_c}{\lambda \sqrt{a}} \]

\[ \Rightarrow N_f = \text{const}(\sigma_{uts}^{-2} - \sigma_{\text{max}}^{-2}) \]
Applying residual strength models to composite laminates

• Strictly speaking, residual strength only makes sense when a single pre-existing crack becomes unstable under applied load, i.e., brittle failure. Griffith criterion then gives $RS = \frac{K_c}{\lambda \sqrt{a}}$

• Residual strength (or strength degradation) can be assumed if general weakening (diffused cracking) occurs. Griffith criterion, however, will not apply. Diffused (distributed) cracking can be described as “damage”, given by a severity index D

• Conditions for Griffith criterion do not exist in unidirectional composites. Also, general weakening does not occur.
Damage in composite laminates

“Damage” here is not quantified, but indicates qualitatively the rate at which the mechanisms are found to increase.

CDS: A state of saturation of matrix cracking, characteristic of a given laminate.

Stage I – Distributed matrix cracking

Define, \( D = \frac{R_0 - R}{R_0 - R_c} \)

Assume, \( \frac{dD}{dN} = k \left( \frac{\sigma_{\text{max}}}{1 - D} \right)^m \)

Integrating and using \( D \) as defined,

\[
(R - R_c)^{m+1} = (R_0 - R_c)^{m+1} \left[ 1 - k(m+1)\sigma_{\text{max}}^m N \right]
\]

\( R_0, R_c, m \) and \( k \) are to be determined from experimental data

Talreja, ZAMM, 2015
Stage II – Localized damage

Define residual strength, $R = \frac{\alpha}{\sqrt{C}}$

where $C$ = “size” of localized damage and $\alpha$ is similar to fracture toughness

Assume, \[ \frac{dC}{dN} = \gamma (C)^{n/2} \]

Integrating and using $R$ as defined,

\[ R_c^{n-2} - R_{n-2}^{n-2} = 2\gamma (n-2)(N - N_c) \]

\[ N_f = N_c + \left( \frac{R_c^{n-2} - \sigma_{\max}^{n-2}}{2\gamma(n-2)} \right) \]

Talreja, ZAMM, 2015
Stiffness degradation based phenomenological models – Stage I

Note: E-modulus decreases mostly in Stage I

Jamison, et al, 1984
Stiffness degradation based phenomenological models – Stage I

From damage mechanics theories,

\[
E = E_0 (1 - D)
\]

=> \( D = 1 - \frac{E}{E_0} \)  Note \( D \neq 1 \)

\[
\frac{dD}{dN} = A \frac{\sigma_{\text{max}}^b}{(1 - D)^c}
\]

Integrating, \( N_c = \frac{(c+1) - \left(\frac{E_0}{E_c}\right)^{c+1}}{(c+1)A\sigma_{\text{max}}^b} \)
Progressive damage models

• These models depend on how “damage” is defined/characterized
• If “damage” is simply a number, e.g. $0 < D < 1$, then the models are phenomenological
• If “damage” is related to the internal energy dissipating mechanisms, and its “progression” (rate of change) is derived from those mechanisms, then the corresponding models are **Progressive damage models** (PDMs)

Many existing models are phenomenological but are sold as PDMs.
Failure in Multi-axial Fatigue

Global loading is MULTIAXIAL
However, Local failure is BIAXIAL
Multiaxial fatigue – failure criteria based approach

Static failure envelope

Assumption: failure envelope in static loading reduces as a function of number of cycles.

Let us examine the failure criteria
Early failure theories for anisotropic solids

Hill (1948), Azzi-Tsai (1965), Tsai-Wu (1971)

• Based on metal yielding
• No consideration of fiber and interface failure
• Assumed interactions of failure modes without any basis in actual physics of failure modes
Background - Metal Plasticity: Stress-based yield criteria

von Mises (1913):

Yielding initiates when the second invariant of the deviatoric stress tensor reaches a critical value (achieved in uniaxial stress state).

In terms of principal stresses:

\[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_1 - \sigma_3)^2 = 2Y^2 \]

\[ Y = \text{yield stress} \]

Or, in terms of stress components:

\[ (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_x - \sigma_z)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2) = 2Y^2 \]

Note: Hencky showed in 1923 that von Mises criterion can be derived from distortion energy.
Hill’s Generalization of von Mises criterion for orthotropic metals

Motivation: Texture (grain stretching in rolling direction) developed in metal sheet forming

Take von Mises criterion for *isotropic* metal yielding:

\[
(s_x - s_y)^2 + (s_y - s_z)^2 + (s_x - s_z)^2 + 6(\tau_{xy}^2 + \tau_{yz}^2 + \tau_{xz}^2) = 2Y^2
\]

MATHEMATICALLY Generalize it to *ANISOTROPIC* yielding:

\[
F(s_y - s_z)^2 + G(s_z - s_x)^2 + H(s_x - s_y)^2 + 2L\tau_{yz}^2 + 2M\tau_{xz}^2 + 2N\tau_{xy}^2 = 1
\]
Closer examination of Hill’s Criterion

Let $X$, $Y$ and $Z$ be yield stresses in $x$, $y$ and $z$ directions and $R$, $S$ and $T$ be yield stresses in shear in the principal planes

$$
\frac{1}{X^2} = G + H, \quad 2F = \frac{1}{Y^2} + \frac{1}{Z^2} - \frac{1}{X^2},
$$

$$
\frac{1}{Y^2} = H + F, \quad 2G = \frac{1}{Z^2} + \frac{1}{X^2} - \frac{1}{Y^2},
$$

$$
\frac{1}{Z^2} = F + G, \quad 2H = \frac{1}{X^2} + \frac{1}{Y^2} - \frac{1}{Z^2},
$$

$$
2L = \frac{1}{R^2}, \quad 2M = \frac{1}{S^2}, \quad 2N = \frac{1}{T^2}.
$$
Closer examination of Hill’s Criterion, contd.

- If $X = Y = Z; R = S = T = Y/2$, then Hill’s criterion reduces to the von Mises criterion.
- Hencky’s interpretation of von Mises criterion as energy of distortion criterion DOES NOT apply to Hill’s criterion.
- The six material constants in Hill’s criterion are all for one physical phenomenon: YIELDING.
Adaptation of Hill’s criterion by Azzi & Tsai (1965)

REPLACE six yield stress constants with six UD composite “strength” constants. Then, in 2-D (in-plane) version, assume $Y = Z$ (transversely isotropic).

$$\left( \frac{\sigma_1}{X} \right)^2 - \left( \frac{\sigma_1 \sigma_2}{X} \right)^2 + \left( \frac{\sigma_2}{Y} \right)^2 + \left( \frac{\tau_{12}}{T} \right)^2 = 1$$

Known as “Tsai-Hill” criterion
Tsai-Hill “Failure” Criterion - Initial Claims of success

Test case: UD composites under uniaxial load

\[
\frac{\cos^4 \theta}{X^2} + \left( \frac{1}{T^2} - \frac{1}{X^2} \right) \cos^2 \theta \sin^2 \theta + \frac{\sin^4 \theta}{Y^2} = \frac{1}{\sigma_x^2}
\]

• Test data seemed to agree well
• For compression, \( X = X' \), \( Y = Y' \), compressive strength values
• **Agreement for general cases not found**

Why is this not the right criterion?
• Failure mechanisms in basic modes (tension, compression, shear) are independent, with different critical states, and cannot therefore be combined to give loss of load carrying capacity (strength).

• The governing failure mechanism for given loading condition must be identified and its criticality must be expressed in terms of combined stresses
Formal criteria with no physical basis

Gol’denblat and Kopnov (1965) proposed a “rather general” SCALAR function of TENSOR Stress Components as

\[(F_{ij}\sigma_{ij})^{\alpha} + (F_{ijkl}\sigma_{ij}\sigma_{kl})^{\beta} + (F_{ijklmn}\sigma_{ij}\sigma_{kl}\sigma_{mn})^{\gamma} = 1\]

Tsai and Wu (1971) “adapted” (simplified) this expression to an orthotropic UD composite, in Voigt notation, as

\[F_p \sigma_p + F_{pq} \sigma_p \sigma_q = 1 \quad p, q = 1, 2, 6\]

Failure surface: Ellipsoid or Elliptical Paraboloid
Hashin’s Failure Theory: The FIRST with some physical considerations

- Hashin (1980) observed that a single smooth (quadratic) failure surface (in $\sigma_{11}$, $\sigma_{22}$, and $\sigma_{12}$) is not supported by two independent failure mechanisms: fiber failure and matrix failure.
- He proposed separate criteria for tensile fiber mode, compressive fiber mode, tensile matrix mode and compressive matrix mode.
- For matrix failure modes he postulated a failure plane inspired by Coulomb’s work in 1776.
Failure envelope:
Single vs. Multiple mechanisms

Interaction, single mechanism

Mechanism 1

Mechanism 2
Hashin (1980) proposal for matrix failure modes

\[
f(\sigma_{nn}, \sigma_{nl}, \sigma_{nt}) = 1
\]

or

\[
g(\sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33}, \theta) = 1
\]
Puck’s Theory (1996):
Same assumptions as in Hashin (1980)

Seven constants needed for inter-fiber (matrix) failure mode
Summary Remarks on all major failure criteria

• The early criteria (Tsai-Hill, Hoffman, Tsai-Wu,..) lack physical basis, and should be regarded as obsolete.
• Hashin, followed by Puck, have physical considerations, but not direct physical evidence to support the hypotheses.

=> Need mechanisms based failure criteria.
Multiaxial fatigue – failure criteria based approach

Assumption: failure envelope in static loading reduces as a function of number of cycles.

Let us examine the failure criteria
Cyclic biaxial stress variation

\( \sigma_x(t) = \sigma_{x,m} + \sigma_{x,a} \sin(\omega \cdot t) \)

\( \sigma_y(t) = \sigma_{y,m} + \sigma_{y,a} \sin(\omega \cdot t - \delta_{y,x}) \)

\( \tau_{xy}(t) = \tau_{xy,m} + \tau_{xy,a} \sin(\omega \cdot t - \delta_{xy,x}) \)

\( \sigma_1(t) = \sigma_{1,m} + \sigma_{1,a} \sin(\omega \cdot t) \)

\( \sigma_2(t) = \sigma_{2,m} + \sigma_{2,a} \sin(\omega \cdot t - \delta_{2,1}) \)

\( \sigma_6(t) = \sigma_{6,m} + \sigma_{6,a} \sin(\omega \cdot t - \delta_{6,1}) \)
Tsai-Hill criterion based prediction of fatigue life

\[
\left( \frac{\sigma_{1,a}}{K_1(N_f)} \right)^2 + \left( \frac{\sigma_{2,a}}{K_2(N_f)} \right)^2 - \frac{\sigma_{1,a} \sigma_{2,a}}{K_1^2(N_f)} + \left( \frac{\sigma_{6,a}}{K_6(N_f)} \right)^2 = 1
\]

where

\[K_i(N_f) = \sigma_{1,A} \left( \frac{N_A}{N_f} \right)^{1/k_i}\]

\(K_i(N_f)\) have to be calculated by using experimental fatigue curves, for \(i = 1, 2\) and 6, for the same R value.
Prediction vs Experimental data using Tsai-Hill criterion: Global coordinates
Prediction vs Experimental data using Tsai-Hill criterion: Local coordinates
Smith-Pascoe Biaxial Fatigue Criterion

Biaxial fatigue of glass-fibre reinforced composite. Part 2: failure criteria for fatigue and fracture”
Proc. of Biaxial and Multiaxial Fatigue - EGF 3 1989

1. Failure by rectilinear cracking
   - governed by strain energy of normal stresses

\[
U_{F,a} = \frac{1}{2} \left\{ \frac{\sigma_{k,a}^2}{E_1} - \left( \frac{\nu_{12}}{E_1} + \frac{\nu_{21}}{E_2} \right) \sigma_{1,a} \sigma_{2,a} + \frac{\sigma_{2,a}^2}{E_1} \right\} = K_{SE} \left( N_f \right) = U_{F,A} \left( \frac{N_A}{N_f} \frac{1}{k} \right)
\]

2. Failure by shear
   - governed by shear stress

\[
\sigma_{6,a} = K_6 \left( N_f \right) = \sigma_{6,A} \left( \frac{N_A}{N_f} \right)^{\frac{1}{k_6}}
\]
Failure when both mechanisms are present, given by

\[ \frac{1}{\sigma_{1,a}^2(N_f)} = \frac{1}{\sigma_{SE,a}^2(N_f)} + \frac{1}{\sigma_{6,a}^2(N_f)} \]

Note:
Needs two fatigue curves for calibration, one in biaxial normal stresses, and the other in pure shear.
Prediction vs Experimental data using the Smith-Pascoe Criterion
Assumption:
If fatigue life is evaluated in one reference direction, it can be obtained in any other direction by a transfer function.

Fatigue life in reference direction $r$ is given by

$$\sigma_{r,\text{max}} = b_r + m_r \log(N_f)$$

And in any generic direction $i$,

$$\sigma_{i,\text{max}} = b_i + m_i \log(N_f)$$
Fewaz-Ellyin Biaxial Fatigue Criterion

Transfer functions:

\[ m_i = f(\lambda_C, \lambda_T, \theta) \cdot g(R) \cdot m_r \]
\[ b_i = f(\lambda_C, \lambda_T, \theta) \cdot b_r \]

where

\[ f(\lambda_C, \lambda_T, \theta) = \frac{\sigma_{x,ult}(\lambda_C, \lambda_T, \theta)}{\sigma_{x,ult}(\lambda_{C,r}, \lambda_{T,r}, \theta_r)} \]
\[ \frac{b_i}{b_r} = \frac{\sigma_{x,ult}(\lambda_C, \lambda_T, \theta)}{\sigma_{x,ult}(\lambda_{C,r}, \lambda_{T,r}, \theta_r)} \]

\[ g(R) = 1 \text{ if } R = R_r \text{ or } R \leq 0 \]
\[ g(R) = \frac{1 - R}{1 - R_r} \text{ if } R > 0, \]
Prediction vs Experimental data
Using the Fewaz-Ellyin Criterion (local coordinates)
Overview of fatigue life prediction process

Initial Deformation Models → Stress Analysis
Stress Analysis
Stress/strain/Temp at Critical Sites → Service Loading

Service Loading → Damage Mechanisms
Damage Mechanisms
Matrix Cracking, Delamination, Viscoelasticity/aging, etc.
Damage Mechanics → Life Prediction
Damage Mechanics
Micro/Meso/Macro Models

Life Prediction → Strength Degradation
Strength Degradation

Stiffness Degradation → Life Prediction
Way forward for a reliable fatigue life prediction

Matrix dominated failure
Fiber dominated failure
Damage Threshold Behavior
Effects of Manufacturing Defects
Failure criteria
Fatigue Life Prediction
Lamination constraints
Interlaminar failure